A Riesz decomposition theorem in W^*-algebras

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To Prof. B. Sz.-Nagy on his 60th birthday

An extension in to W^*-algebras of the Riesz theory ([7]) of compact operators was initiated and developped by M. BREUER in [2]. Here the ideal of compact operators is replaced by the norm closed two-sided ideal generated by all finite projections. A problem raised by M. Breuer was solved by V. I. OVTSHINNIKOV in the case of von Neumann algebras on a separable Hilbert spaces ([6]). Our purpose is to solve the Breuer problem in the general case.

For notions and knowledges about W^*-algebras we send the reader to [5].

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In the first section we prove some technical results. Lemma 3 in the second section solves the conjecture of Breuer and, together with Breuer's results, it allows us to formulate an extension of the Riesz decomposition theorem for compact operators.

1. The two-sided ideal associated to a semi-finite trace

Let M be a W^*-algebra and M^+ its positive part. A mapping

$$\varphi : M^+ \to R^+ \cup \{+\infty\}$$

is called a *normal trace* on M if:

1. $\varphi(x+y) = \varphi(x)+\varphi(y), \quad x, y \in M^+,$
2. $\varphi(\lambda x) = \lambda\varphi(x), \quad x \in M^+, \ \lambda \in R^+,$
3. $\varphi(u^* x u) = \varphi(x), \quad x \in M^+, \ u \in M$ unitary.
4. $\varphi(x) = \sup \varphi(x_\iota)$ for every increasingly filtered family (x_ι) in M^+ with $x = \sup x_\iota$.

We denote by $\mathfrak{M}_\varphi$ the linear span of $\{x : x \in M^+, \varphi(x) < +\infty\}$. $\mathfrak{M}_\varphi$ is a two sided ideal, called the two-sided ideal associated to φ. φ can be extended to a positive linear form on $\mathfrak{M}_\varphi$. If $a \in \mathfrak{M}_\varphi$, then $x \mapsto \varphi(ax) = \varphi(xa)$ is a w-continuous linear form on M, where the w-topology is the weak topology defined by the predual of M. φ is said to be *semi-finite* if:

5. $\mathfrak{M}_\varphi$ is w-dense in M.

Finally, a normal trace φ on M is called *faithful* if

6. $x \in M^+$, $\varphi(x) = 0 \Rightarrow x = 0$.

If there exists a faithful semi-finite normal trace on M, then M is called semi-finite.

A projection $e \in M$ is called finite if every projection $f \in M$ $f \leqq e$, $f \sim e$, equals e. The norm-closed linear span $\mathscr{I}$ of all finite projections $e \in M$ is a two-sided ideal. Every projection $e \in \mathscr{I}$ is finite.

Let M be a W^*-algebra. For every set $S \subset M$ we denote by $\langle S \rangle$ the set of all elements $\sum_{\iota \in I} x_\iota z_\iota$, where $x_\iota \in S$ and (z_ι) is a family of orthogonal central projections with $\sum_{\iota \in I} z_\iota = 1$. If M is semi-finite, φ is a faithful normal semi-finite trace on M and e is a projection in $\mathfrak{M}_\varphi$, then e is finite. So, denoting by $\overline{\mathfrak{M}}_\varphi$ the norm-closure of $\mathfrak{M}_\varphi$, we have $\langle \overline{\mathfrak{M}}_\varphi \rangle \subset \mathscr{I}$.

Lemma 1. *Let M be a W^*-algebra, φ a normal semi-finite trace on M and $e \in M$ a finite projection. Then $e \in \langle \mathfrak{M}_\varphi \rangle$.*

Proof. Let $(z_\iota)_{\iota \in I}$ be a maximal family of orthogonal central projections such that $ez_\iota \in \mathfrak{M}_\varphi$ for all $\iota \in I$. Suppose that $z_0 = 1 - \sum_{\iota \in I} z_\iota \neq 0$.

Since $\mathfrak{M}_\varphi$ is w-dense in M, there exists a non-zero projection $f \in \mathfrak{M}_\varphi z_0$. By the comparison theorem there exists a central projection $z_1 \leqq z_0$, such that

$$fz_1 \prec ez_1, \qquad f(z_0 - z_1) \succ e(z_0 - z_1).$$

The maximality of $(z_\iota)_{\iota \in I}$ implies $z_0 - z_1 = 0$. So

$$f = fz_0 \prec ez_0$$

that is, for some projection $f_1 \in M$,

$$f \sim f_1 \leqq ez_0.$$

Using again the comparison theorem, there exists a central projection $z_2 \leqq z_0$, such that

$$fz_2 \prec (ez_0 - f_1)z_2, \qquad f(z_0 - z_2) \succ (ez_0 - f_1)(z_0 - z_2).$$

Again, by the maximality of $(z_\iota)_{\iota \in I}$ we have $z_0 - z_2 = 0$. Hence

$$f = fz_0 \prec ez_0 - f_1$$

that is, for some projection $f_2 \in M$

$$f \sim f_2 \leqq ez_0 - f_1.$$

By induction, we obtain a sequence (f_n) of orthogonal projections such that, for every n, $f_n \sim f$ and $f_n \leqq ez_0$. This contradicts the finiteness of e.

Consequently $z_0 = 0$ that is $e \in \langle \mathfrak{M}_\varphi \rangle$.

Lemma 2. *Let $\mathfrak{N}$ be a norm-closed two-sided ideal in a W^*-algebra M. Then $\langle \mathfrak{N} \rangle$ is a norm-closed two-sided ideal.*

Proof. Obviously, $\langle \mathfrak{N} \rangle$ is a two-sided ideal.

Let (x_n) be a sequence in $\langle \mathfrak{N} \rangle$ convergent in the norm-topology to $x \in M$. For every n there exists a family $(z_n^\iota)_{\iota \in I_n}$ of orthogonal central projections, such that

$$x_n z_n^\iota \in \mathfrak{N}, \iota \in I_n, \quad \text{and} \quad \sum_{\iota \in I_n} z_n^\iota = 1.$$

Denote by Ω the maximal ideal space of the center of M. Then Ω is a hyperstonean space (see [3]). For every n, $\Omega_n = \bigcup_{\iota \in I_n} (z_n^\iota)^{-1}(1)$ is an open dense set, and $\bigcap_{n=1}^{\infty} \Omega_n$ contains an open dense set Ω_0. Now it is easy to see, that there exists a family $(K_\iota)_{\iota \in I}$ of mutually disjoint open and closed subsets of Ω_0 such that $\bigcup_{\iota \in I} K_\iota$ is dense in Ω. Denote by z_ι the characteristic function of K_ι. Then $(z_\iota)_{\iota \in I}$ is a family of orthogonal central projections with $\sum_{\iota \in I} z_\iota = 1$. For every $\iota_0 \in I$ and for every n, $\left((z_n^\iota)^{-1}(1)\right)_{\iota \in I}$ is an open convering of $(z_{\iota_0})^{-1}(1) = K_{\iota_0}$. Hence there exist $\iota_1, \dots, \iota_{k_n} \in I$ such that

$$(z_{\iota_0})^{-1}(1) \subset \bigcup_{j=1}^{k_n} (z_n^{\iota_j})^{-1}(1), \text{ that is } z_{\iota_0} \leqq \sum_{j=1}^{k_n} z_n^{\iota_j}.$$

Consequently, for every $\iota \in I_n$ and for every n we have $x_n z_\iota \in \mathfrak{N}$. $\mathfrak{N}$ being norm-closed, we obtain $x z_\iota \in \mathfrak{N}$, $\iota \in I$. So $x \in \langle \mathfrak{N} \rangle$.

In [1], a norm-closed two-sided ideal $\mathfrak{M}$ in M such that $\langle \mathfrak{M} \rangle = \mathfrak{M}$ is called continuous. Examples of continuous ideals are the above defined ideal $\mathscr{I}$ and the strong radical. Lemma 2 states that the continuous hull of $\mathfrak{N}$ is $\langle \mathfrak{N} \rangle$.

Theorem 1. *Let M be a W^*-algebra and φ a normal semi-finite trace on M. Then $\mathscr{I} \subset \langle \overline{\mathfrak{M}_\varphi} \rangle$.*

Proof. Let $\mathscr{I}_0$ be the two-sided ideal generated by all finite projections $e \in M$. By Lemma 1, $\mathscr{I}_0 \subset \langle \mathfrak{M}_\varphi \rangle \subset \langle \overline{\mathfrak{M}_\varphi} \rangle$. On the other hand, by Lemma 2 $\langle \overline{\mathfrak{M}_\varphi} \rangle$ is norm-closed. Hence the norm-closure $\mathscr{I}$ of $\mathscr{I}_0$ is included in $\langle \overline{\mathfrak{M}_\varphi} \rangle$.

Using Theorem 1 and the remark preceding Lemma 1, we obtain:

Corollary. *If M is a semi-finite W^*-algebra and φ is a faithful normal semi-finite trace on M, then $\mathscr{I} = \langle \overline{\mathfrak{M}_\varphi} \rangle$.*

We need also the following result:

Theorem 2. *Let M be a semi-finite W^*-algebra, φ a faithful normal semi-finite trace on M, and N a W^*-subalgebra of M such that $N \cap \mathfrak{M}_\varphi$ is w-dense in N. Then there exists a positive linear map $P: M \to N$ such that*

1) $P(1)$ *is the unit of* N.
2) $P(y_1 x y_2) = y_1 P(x) y_2,\ x \in M,\ y_1, y_2 \in N.$
3) $P\mathfrak{M}_\varphi \subset N \cap \mathfrak{M}_\varphi$.
4) $\varphi(P(x)) = \varphi(P(1)x),\ x \in \mathfrak{M}_\varphi$.

For the proof of Theorem 2 we send the reader to [4] (see also [5] chap. III, § 5, exercise 9. f).

2. The main result

Let M be a W^*-algebra and $a \in M$. For every integer $n \geqq 1$ we denote by $e_n(a)$ the greatest projection $e \in M$ such that $(1-a)^n e = 0$. We denote also $e_\infty(a) = \sup_n e_n(a)$.

Now we suppose that a belongs to the norm-closed two-sided ideal $\mathscr{I}$ generated by the finite projections. Since for every integer $n \geqq 1$ the equality $e_n(a) = \left(1-(1-a)^n\right)e_n(a)$ holds, we have $e_n(a) \in \mathscr{I}$. Consequently $e_n(a)$ are finite projections. This statement is sharpened by the following

Lemma 3. *If $a \in \mathscr{I}$ then $e_\infty(a)$ is finite.*

Proof. First, we remark that $e_\infty(a)$ belongs to the w-closure of $\mathscr{I}$. So we can suppose that M is semi-finite. Let φ be a faithful normal semi-finite trace on M.

By Theorem 1, there exists a family $(z_\iota)_{\iota \in I}$ of orthogonal central projections in M with $\sum_{\iota \in I} z_\iota = 1$ such that $az_\iota \in \overline{\mathfrak{M}}_\varphi$ for all ι. Since $e_\infty(az_\iota) = e_\infty(a)z_\iota$ and it is sufficient to prove that every $e_\infty(a)z_\iota$ is finite, we can suppose that $a \in \overline{\mathfrak{M}}_\varphi$.

Let N be the W^*-subalgebra of M generated by the projections $e_n(a)$, $1 \leqq n < \infty$. Then N is commutative and its unit is $e_\infty(a)$. Since $e_n(a) = \left(1-(1-a)^n\right)e_n(a)$, $1 \leqq n < \infty$, the projections $e_n(a)$ belong to $\overline{\mathfrak{M}}_\varphi$. By [5], chap. I, § 1, exercise 6, they belong even to $\mathfrak{M}_\varphi$. Hence $N \cap \mathfrak{M}_\varphi$ is w-dense in N.

By Theorem 2 there exists a positive linear map $P: M \to N$ such that

1) $P(1) = e_\infty(a)$,
2) $P(y_1 x y_2) = y_1 P(x) y_2,\quad x \in M,\ y_1, y_2 \in N,$
3) $P\mathfrak{M}_\varphi \subset N \cap \mathfrak{M}_\varphi$.

For every $2 \leqq n < \infty$ we have

$$(1-a)^{n-1}\left(1-e_{n-1}(a)\right)(1-a)e_n(a) = 0.$$

Since $1-e_{n-1}(a)$ is the right support of $(1-a)^{n-1}$, it follows that

$$\left(1-e_{n-1}(a)\right)(1-a)e_n(a)=0$$

Consequently

$$P(1-a)\left(e_n(a)-e_{n-1}(a)\right)=\left(e_\infty(a)-e_{n-1}(a)P(1-a)e_n(a)=\right.$$
$$=P\left((e_\infty(a)-e_{n-1}(a))(1-a)e_n(a)\right)=0.$$

This equality implies

$$P(1-a)e_1(a)=P(1-a)e_2(a)=\cdots=P(1-a)e_\infty(a)=P(1-a).$$

Since

$$P(1-a)e_1(a)=P\left((1-a)e_1(a)\right)=0,$$

we conclude

$$P(1-a)=0.$$

Now we have

$$e_\infty(a)=P(1)=P(a)\in\overline{N\cap\mathfrak{M}_\varphi}.$$

By [5], chap. I, § 1, exercise 6, $e_\infty(a)\in N\cap\mathfrak{M}_\varphi\subset\mathfrak{M}_\varphi$. Hence $e_\infty(a)$ is a finite projection.

For every $a\in M$ and for every integer $n\geqq 1$ we denote by $f_n(a)$ the left support of $(1-a)^n$. Put $f_\infty(a)=\inf_n f_n(a)$. Lemma 3 and the results of BREUER ([2], Theorems 1 and 2) imply the following theorem, an analogue of the Riesz decomposition theorem in the theory of compact operators:

Theorem 3. *Let M be a W^*-algebra, $\mathcal{I}$ the norm-closed two-sided ideal generated by all finite projections in M, and a an element of $\mathcal{I}$. Let the projections $e_n(a)$, $f_n(a)$, be defined as above. Then*

(i) *$e_\infty(a)$ is a finite projection;*
(ii) *for every $1\leqq n<\infty$ there exist $x_n\in M$ and a finite projection $p_n\in M$ such that $(1-a)^n x_n=1-p_n$;*
(iii) *$ae_\infty(a)=e_\infty(a)ae_\infty(a)$ and $af_\infty(a)=f_\infty(a)af_\infty(a)$;*
(iv) *$e_\infty(a)\wedge f_\infty(a)=0$ and $e_\infty(a)\vee f_\infty(a)=1$;*
(v) *for every $1\leqq n\leqq\infty$ we have $e_n(a)\sim e_n(a^*)=1-f_n(a)$.*

References

[1] C. Apostol and L. Zsidó, Ideals in W^*-algebras and the function η of A. Brown and C. Pearcy, *Revue Roum. Math. Pures et Appl.* (to appear).
[2] M. Breuer, Fredholm theories in von Neumann algebras I, *Math. Annalen*, **178** (1968), 243—254.
[3] J. Dixmier, Sur certains espaces considérés par M. H. Stone, *Summa Brasil. Math.*, **2** (1951), 151—182.
[4] J. Dixmier, Formes linéaires sur un anneau d'opérateurs. *Bull. Soc. Math. France*, **81** (1953), 9—39.
[5] J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien (Algèbres de von Neumann)*, deuxième édition, Gauthier-Villars (Paris, 1969).
[6] V. I. Ovtshinnikov, On completely continuous operators relative to a von Neumann algebra (in Russian), *Funkcionalniĭ anal. i pril.*, **6** (1972), 37—40.
[7] F. Riesz et B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, 3^e éd. Gauthier-Villars et Akadémiai Kiadó (Paris—Budapest, 1955).

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